Section 5. Search for new ground in pedagogy

OBSERVING TEACHERS: THE MATHEMATICS PEDAGOGY OF QUEBEC FRANCOPHONE AND ANGLOPHONE TEACHERS

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Abstract: *This text presents preliminary results of a study comparing mathematics teaching pedagogies of Quebec Francophone and Anglophone Mathematics teachers. This study is part of a new nationwide study on middle school mathematics pedagogies conducted in four different regions in Canada, where the main objective is to describe regional difference in mathematics teaching and underlying pedagogies in Canada, and to relate these differences in student achievement in mathematics. Four Anglophone and four Francophone Quebec teachers engaged in focus groups based on three types of lessons they each previously recorded: typical, exemplary and introductory. The focus groups discussions were analyzed and compared to reveal aspects of the teachers' pedagogies.*

Context of the Study

The large scale international assessments conducted by PISA have revealed that there is a considerable range of student achievement in mathematics across Canada. When compared to international results, some Canadian provinces rank among the top countries, while other provinces are significantly below the Canadian average. A number of factors have been claimed to contribute to these differences including curriculum, students' genders, attitudes, beliefs, aspirations, and time spent working outside school, parents' education, involvement and socio-economic status, other aspects of the home environment, and school resources (Anderson et al., 2006; Beaton & O'Dwyer, 2002; Schmidt, et al., 2001; Wilkins, Zembylas, & Travers, 2002). Those factors might have a strong influence on students' learning, but other factors might explain the difference of students' achievement in mathematics. For instance, the test sheds light on differences in language, curriculum, and culture as factors to consider when interpreting the results. But there is no insight into differences in teaching pedagogies. Teaching pedagogies imply the use of different approaches when teaching. Even if the mathematical content is the same, the pedagogies used by teachers might not be the same. So far, no comparison of teaching and pedagogy between regions of Canada has been made, in spite of the noticeable differences in student achievement between regions. The results analyzed here come from a Canada-wide research program intended to address this lack of knowledge (see http://www.acadiau.ca/~dreid/OT/index.html).

7 The research on which this article is based is a collaborative research project and the collection and analysis of data and the writing of this article have involved the efforts of many people in addition to those listed above. For more information see http://www.acadiau.ca/~dreid/OT/. The research supported by the Social Sciences and Humanities Research Council of Canada (SSHRC grant #410-2011-1074).

The objective of this research project is to describe regional differences in mathematics teaching and underlying pedagogies in Canada, and to relate these differences in student achievement in mathematics. The guiding questions for this study are: How do pedagogies in middle school mathematics in regions of Canada differ? How are these differences related to differences in average achievement and the range of achievement in regions of Canada?

This research report presents preliminary results of this study done in the province of Quebec, one of the regions chosen in this nationwide study. Here we focus on comparing the teaching practices and pedagogies between Quebec Francophone and Anglophone teachers. Our main research question is: How are the mathematics teaching pedagogies similar and/or different between Quebec Francophone Anglophone teachers?

Theoretical Framework

As our objectives are focused on pedagogy, it is important to clarify what this word means to us. We are referring to what Tobin et al. (2009) call "'implicit cultural practices' of teachers, by which we mean practices that though not taught explicitly in schools of education or written down in textbooks reflect an implicit cultural logic" (p. 19). As they note this concept is related to what Anderson-Levitt (2002, p. 109) refers to as "knowledge in practice" and "embodied knowledge" (p. 8) which refer to "professional knowledge rooted in national classroom cultures" (p. 109) as well as Bruner's (1996) concept of "folk pedagogy": "taken-for-granted practices that emerge from embedded cultural beliefs about how children learn and how teachers should 'teach'" (p. 46).

One aspect of pedagogy is the connection, or lack of connection the teacher sees between mathematics and other disciples. All teachers are complex personalities that go beyond simply being teachers of their subjects. They may also be musicians, avid readers, DIY home renovators, or practitioners of a myriad of other activities. The degree to which and the ways in which these other aspects of teachers' personalities affects their teaching, have thus far been researched only at the level of easily observable changes to teaching materials. More subtle influences at the level of pedagogy have not been researched, and our work seeks to address this omission. Savard & Manuel (2015) provides on example, in which the links teachers made between statistics and other disciplines was researched.

Method

The methodology for this research is an enactivist one (Reid 1996). Four Francophone and four Anglophone grade 7 and/or 8 (known as secondary 1 and 2 in Quebec) mathematics teachers participated in this study. The study was divided into two phases. During the first phase, all the teachers were recorded teaching three mathematics lessons: one that they felt was a typical lesson in their classrooms; one that they felt was exemplary; and one introductory lesson on a topic related to fractions. Each lesson recorded was then edited by members of the research team and 20 to 25 minutes of each lesson where kept. The content of the edited videos focussed on the different practices the teachers used in their lessons.

The second phase consisted of data collection via focus group discussions. The four Francophone teachers met with the research team for three focus group sessions. Three of the four Anglophone teachers met with the research team for three focus groups sessions at a different time. The fourth Anglophone teacher had to leave the project because she was on sick leave when the focus groups were held. She, however, gave us permission to use her videos. The first focus group was set up to discuss the typical lessons. The edited video of each teacher's lesson was shown and then discussed. Before viewing a video, the teacher presented the activity she did and her learning goals and answered questions from the group. After the video, a discussion on the practices she used occurred. After all the videos were shown, the teachers had the task of creating a set of criteria that defined a typical lesson and had to choose one of the four videos as the best typical lesson based on the criteria they set. The same process was repeated for the introductory lesson on a topic related to fractions and for the exemplary lesson. The Francophone teachers had a hard time setting up a list of criteria. They wanted to view and discuss all the videos before coming up with their criteria. A fourth focus group session was added so they could then create criteria for the practices found in all three types of lessons and then select a video for each type of lesson. The research team facilitated the discussions and the group of teachers came up with a table defining lessons that are exemplary, lessons that are typical, and what practices should be in lessons introducing fractions. The Anglophone teachers preferred to discuss the criteria before watching the videos but they refined their criteria throughout the process. In both groups, it was hard to come up with only one video for each type of lessons: all the teachers wanted to have parts from several videos. For further discussion of the use of video in our research, see Reid, Simmt, Savard, Suurtamm, Manuel, Lin, Quigley, & Knipping (2015).

All focus groups were video recorded and then transcribed using pseudonyms. A thematic approach was used to analyze the data (Butler-Kisber, 2010). A comparison was done for each type of lesson used in this study between the Francophone and Anglophone teachers by analyzing the practices mentioned in the corpus for each type of lesson based on the criterion from both the Francophone and Anglophone teachers (Butler-Kisber, 2010).

Results

Typical lesson

According to the Francophone teachers, a typical lesson is one that is done most of the time and follows a certain routine or a functioning method that was established at the beginning of the school year. There is also a particular sequence to a typical lesson that consists of: a follow-up on the homework, an explanation of the new content, and exercises. One of the teachers summarizes by saying, "In reality, a typical lesson, this is what we do 70% of the time: correcting homework, then present the new content and then students do some work… There is a particular routine and the students are used to it" (all quotations of Francophone teachers have been translated from French by the authors). For the Francophone teachers, the sequence also includes mini quizzes (either formative or summative) at the beginning of the lesson that mostly focus on mental math problems and short answer questions related to prior concepts learned during the year. Two of the teacher used that practice. One focused mainly on mental math while they poses different types of questions (mental math, review prior concepts and operations, short word problems). One of them takes this practice to heart by saying: "I do this because I want my students to find different strategies to solve different problems with whole numbers, fractions and percentages. I give some mental math problems and then we discuss each of them and I have the students share their strategies". The teachers also put an emphasis on using the proper mathematical vocabulary and having a synthesis of the content, which usually consists of taking or creating notes related to the content: "For me, we should always have a synthesis in not only a introductory lesson but also in a typical lesson. We have to let the students know where these mathematical ideas are inserted in the learning process. It should be some kind of conclusion that we do at the end". In addition, although these teachers believe in maintaining a particular sequence in a typical lesson, they also mentioned the importance of questioning students during the explanations or during the correction of the quizzes in order to reveal multiple strategies used in solving problems. For those who have access to technology, using multiple applets to present and work on the mathematical content is also part of their typical practices.

The Anglophone teachers also discussed the norms in class. They mentioned that in a typical lesson, expectations are set. They also talked about the classroom routine, which includes reviewing the homework, presenting new content and then assignments. One of the teachers said: “It is the norm. We should not be expecting anything less than this”. The teachers also described the typical lesson as a talk and chalk lecture style of lesson where the questions asked and the brainstorming sessions are guided by the teachers and students respond to the teacher’s prompt. The Anglophone teachers did put an emphasis on the type of students they have when describing the pedagogies in a typical lesson. They mentioned that: “when groups are weaker, there will be more modeling and prompting, but when they have stronger groups, they can use more student-centered approaches to teaching”. Finally, as for work sessions, the teachers mentioned that there could be pair work during these types of lessons.

Exemplary lesson

The Francophone teachers believe that an exemplary lesson is more out of the ordinary and requires more planning, preparation and originality. “It is what we do less often”, said one of the teachers present. In addition, for it to be successful, the teacher must know the curriculum very well. A teacher added: “I can’t see myself doing exemplary lessons at the beginning of the school year and if it’s the first time I teach this course because I am not comfortable with the curriculum or I don’t know my students well enough during that time”. In this type of lesson, the teachers mentioned that they must give more control to the students during the activities and the classroom management is different. Although students have more control in these types of activities, the teachers also mentioned the importance of guiding the students throughout the process. As for the activities, the teachers believe that an exemplary lesson is centered on an initiating or consolidating task that involves pleasure, manipulation, exploration and collaborative work. One of the teachers summarized these practices by saying: “For me an exemplary lesson, we give the control to students and they are active. They experiment, they manipulate, they discover, etc.”. Even though more control is given to the students, the Francophone teachers added that it is important to set out the rules and expectations at the beginning of the task. The teachers also mentioned the importance of a synthesis at the end of the activity, but this one is different compared to the typical lesson. In this synthesis, the teachers must focus on the initial goals of the activity, different mathematical representations of mathematical ideas, and proper mathematical vocabulary. When discussing about the strength of the students mathematically, one teacher was a bit hesitant about exemplary lesson and seemed to associate this type of lesson with stronger students. She said:

> When I think of the different groups I have, I can’t see myself making exemplary lessons with my weaker students. I have some that still can’t differentiate the numerator and the denominator of a fraction. With stronger groups, I have no problems with exemplary lessons, but not with weaker students. I find sometimes that the groups we have are too heterogeneous.

Although the teachers agreed that exemplary lessons are not done often, one of them mentioned that since most of the content taught in grade 7 mathematics are concepts that the students learned during elementary school, it could be an opportunity to make more exemplary lessons. “I don’t know why we don’t do that many exemplary lessons when they have learned the content we are teaching. I think we should” .

For the Anglophone teachers, an exemplary lesson is one that offers opportunities for students and even teachers to learn and exhibits best classroom practices such as: creating a more student-centered classroom management dynamic in classrooms (students are engaged and involved in a task), questioning (eliciting higher order thinking questions), emphasizing mathematical vocabulary, representing mathematical understandings in various ways, and making connections to other concepts. "Teachers take risks and try new things … You can connect yourself with the real world with this and it allows the teacher and the student to deviate from the lesson plan at times", mentioned one of the teachers. They also mentioned that in this type of lesson, students are exposed to tasks where they must answer "how and why" questions and where expectations are set and the mathematical content is conveyed in them. Also, in such tasks, students explain not only the procedures of their work, but must also justify their choices and their understandings. A teacher summarized that idea by saying: "In this type of lessons, students must not only explain the procedures of their work but also explain the why and their understandings". Thus, according to the Anglophone teachers, exemplary lessons provide opportunities for students to share their mistakes or their alternative concepts and approaches. One teacher was clear about this point and said: "we celebrate mistakes in these lessons". They also added that these types of lessons would be able to reach most students. In addition, the teachers suggested that an exemplary lesson establishes a safe environment for students to participate in the classroom. However, similar to the Francophone teachers, they also mentioned the importance of knowing and being comfortable with the curriculum. They added that it is important to know the content at other school levels in order to have a vision of continuity in the content across levels, to prepare students to go further and deeper into the content, and also to be aware of the students' prior knowledge.

Introductory lesson

Similar to the typical and the exemplary lesson, the Francophone teachers emphasized the importance of mathematical vocabulary in introductory lessons; they mentioned it as an important practice for that type of lessons. The teachers also talked about the importance of activating prior knowledge in introductory lessons, which was a practice not emphasized in the other types of lesson. In addition, in introductory lessons, the Francophone teachers also discussed the importance of varying the representations of mathematical ideas. For instance, for fractions, a teacher commented that in some contexts, it would be more beneficial to use rectangular shapes to compare fractions instead of circular representations, although both are area models. They were impressed by how a student understood that when comparing unit fractions, a part is smaller when the denominator is greater when the teacher drew rectangular pies instead of circular ones. "I always use circular pies or pizzas to represent my fractions. And I just realized in Valérie's video how more clear it was using a rectangle instead of a circle" mentioned one of them. In addition, after seeing examples in a video where one teacher used the number line, others realized that they should use it more often to represent fraction concepts. For instance one mentioned: "After seeing Sophie's video, I realize that I never use the number line to represent fractions. I need to use it more often because it could help some students" . The various representations seemed to have been an eye opener for most of the teachers here. Finally, the Francophone teachers also mentioned the importance of synthesizing the content. They mentioned that it is important to situate the concepts all through the teaching sequence.

The Anglophone teachers created only two videos of introductory lessons. After looking at both of them, the decision was clear for them which should be the one selected. Because of this, the Anglophone teachers did not feel the need to create a list of practices that should be included for an introductory lesson.

Discussion

While these results only allow us to compare pedagogies of teachers in two language groups in a single region, there are already interesting similarities and differences apparent.

The similarities include the structure of the typical lesson, an emphasis on mathematical vocabulary, the use of multiple representations (at least in exemplary lessons) and a belief that a high level of knowledge of the curriculum is important in planning exemplary lessons. The fact that these similarities exist support, in a very preliminary way, an assumption of the overall research project: that teachers working in a regional and/or linguistic community develop a shared pedagogy. As data from other regions is analyzed and compared with the Québec data it will be possible to identify more precisely the extent to which the similarities observed here between the two linguistic groups in Quebec are regional or pan-Canadian.

As researchers with experience in teacher education in a number of regions, it seems likely that the structure of the typical lesson and the focus on multiple representations are not unique to Quebec, although their prevalence may vary regionally. It is possible that the emphasis on vocabulary is a reflection of the prominent role given vocabulary in the Quebec curriculum documents, in which case we would expect less emphasis on vocabulary in other regions.

While teachers in Quebec follow the same curriculum documents, and in many cases use textbooks written specifically for use in the province, the presence of two language communities in the province means that within the region there could well be two distinct pedagogies divided on linguistic lines. More data is needed to establish this, but the differences between the two focus groups are suggestive. These differences include differences related to questioning, synthesis, and attention to student ability.

There is more emphasis on questioning for evaluation in the Anglophone group’s discussion of the typical lesson, compared to an emphasis on questioning as a means of revealing contrasting strategies in the Francophone group. The use of questioning to create such learning opportunities was mentioned by the Anglophone teachers more in the context of an exemplary lesson. Questions that ask for justifications, explaining how and why, were also mentioned by the Anglophone teachers in the context of the exemplary lesson, but not by the Francophone teachers (see Reid, Savard, Manuel, & Lin 2015 for further discussion of this point).

The Francophone teachers mentioned the importance of a synthesis in a lesson, although this plays a different role in the typical and the exemplary lesson. The Anglophone teachers, in contrast, did not mention any need for a synthesis of the content or of the lesson goals.

The Anglophone teachers commented that the perceived level of the students would affect the nature of a typical lesson. If the students are stronger, the typical lesson would have more features of the exemplary lesson. The Anglophone teachers also mentioned that the exemplary lesson would be able to reach most students, implying that different student levels are to be expected and affect teaching. In contrast the Francophone teachers seemed to be more hesitant of using exemplary lessons with weaker groups. Therefore, this was not found in their lists of practices. They mostly put the emphasis on teacher planning and preparation for the success of a lesson.

These three differences suggest that there may be two distinct pedagogies present in Quebec, divided along linguistic lines. This could partially account for the different levels of success of Francophone and Anglophone students on large scale mathematics assessments reported by HRSDC, CMEC & SC (2007).

Conclusion

The results presented here only allow for very preliminary conclusions regarding the differences in regional pedagogies on Canada and their connection to student achievement. There are indications, however, of differences in pedagogy between the two linguistic communities in Quebec.

In the next phases of the research the discussions on the focus groups in other regions will be analyzed and compared with the discussions in the Quebec group. Some of these results have been presented elsewhere (Reid, et al. 2015; Lazarus, Kwai, & Suurtamm 2014). Additional data will be generated when the focus groups view and discuss the videos selected by the groups in other regions. The teachers' observations of similarities and differences in the teaching in other regions will provide further insight into their pedagogies. One focus in this phase will be how the degree to which and the ways in which teachers see connections between mathematics and other disciples affects their teaching.

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PUZZLES AND DISSECTIONS

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Abstract: *We will come across different aspects of equivalence by dissection: Variations on the theorem of Pythagoras, differences between methods and creativity, symmetry, optimizing, rational and irrational rectangles, color and esthetics.*

Introduction

We will discuss different aspects of area equivalence. In general we can prove the equivalence of the areas of two figures either by calculation or by visual thinking.
Examples: The shaded square and the shaded rectangle in the figure 1a have obviously (visual thinking by dissection into unit squares) the same area. On the other side everybody would agree with the calculated equivalence $2 \times 2 = 1 \times 4$. In the example of the figure 1b (unit circle and rectangle with sides 1 and π) there is no common dissection to prove the area equivalence. We have to apply the usual formulas.

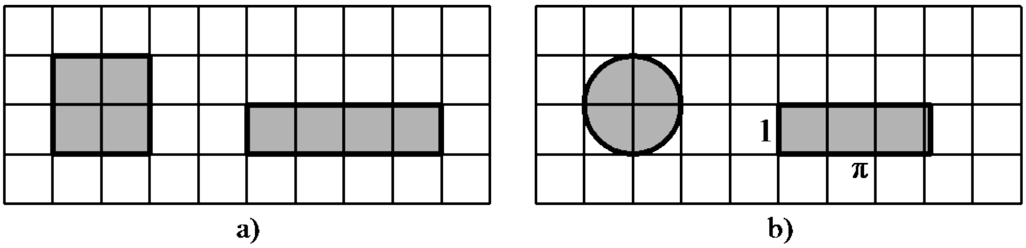

Figure 1. Square and rectangle. Circle and rectangle

In the modification of the unit circle to the regular dodecagon (Fig. 2) we may find the area both by calculation and by dissection. The calculation is easy:

$$A_{\text{Dodecagon}} = 12 \cdot \tfrac{1}{2} \cdot \underbrace{\sin(30^\circ)}_{\frac{1}{2}} = 3$$

The dissection is a challenge to find a beautiful solution (Wells, 1991), (Pandi and Walser, 2012).

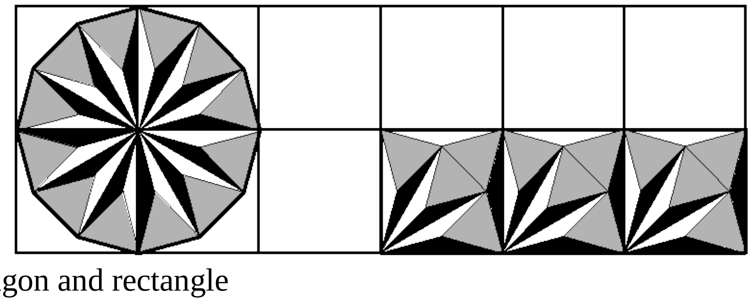

Figure 2. Dodecagon and rectangle

In the example of figure 3 it is not possible to calculate the area of the figure 3b. But we can dissect the figure 3b and rearrange the parts such that the equivalence to the parallelogram is obvious.

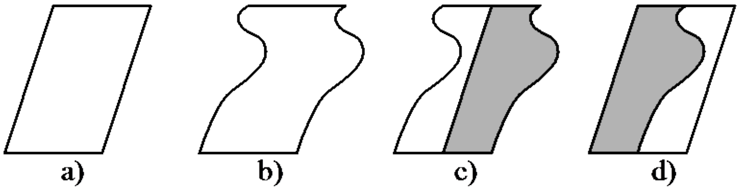

Figure 3. Equivalent areas

In the plane two polygons of equal area have always a common dissection. This is a theorem due to Wallace, Bolyai (the father of Janos Bolyai), and Gerwien (1832).

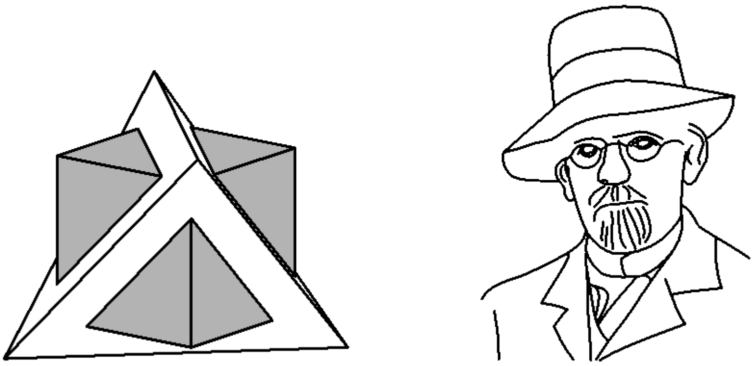

Figure 4. Cube and tetrahedron. David Hilbert, 1862-1943

Now we compare a cube to a tetrahedron with the same volume (Fig. 4). The symmetry group of the tetrahedron is a subgroup of the symmetry group of the cube.

Note that the symmetry group of the equilateral triangle is not a subgroup of the symmetry group of the square. Nevertheless the square and the equilateral triangle have common dissections (Fig. 8 and 9).

In 1900 David Hilbert asked in his third problem of his famous 23 problems the analogue question for polyhedral in space, i.e. whether or not a tetrahedron and a cube of the same volume have a common dissection.

This third problem was the first to be solved. Already in the same year 1900 Dehn could prove that the cube and the tetrahedron of the same volume have no common dissection. In 1903 Kagan gave another proof, and in 1954 Hadwiger generalized into higher dimensions. About Hilbert's third problem see (Aigner and Ziegler, 2009), (Benko, 2007), (Boltianskii, 1978), (Dehn, 1900), (Dehn, 1902), (Hadwiger, 1949/50), (Hadwiger, 1954), (Kagan, 1903), and (Wittmann, 2012).

In the following sections we will remain in the two-dimensional plane and study some equal-area-theorems as for example the theorem of Pythagoras.

Dissection proofs of the theorem of Pythagoras

Figure 5a depicts a classic dissection proof of the theorem of Pythagoras.

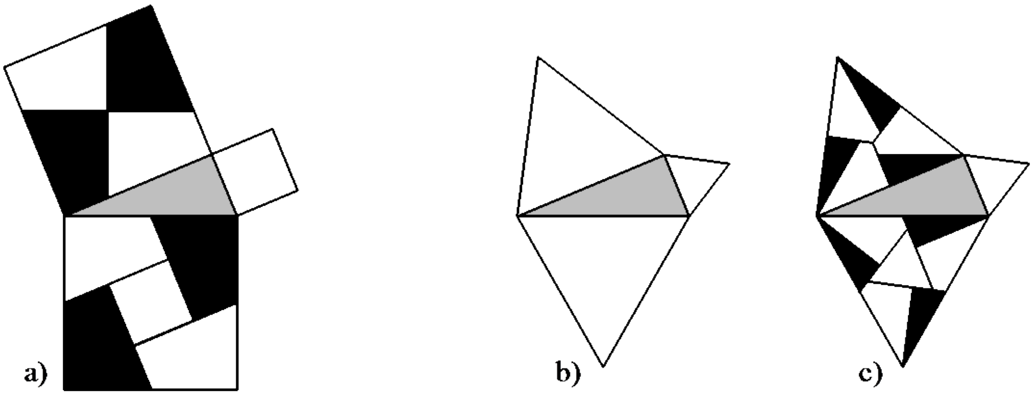

Figure 5. Classic dissection. Equilateral triangles

But the theorem of Pythagoras holds also with other polygons instead of squares, for example with equilateral triangles (Fig. 5b). Figure 5c shows a dissection for this case. This kind of dissection works as well with other regular polygons (Fig. 6). The small black right triangles are always the same. They are similar to the original (shaded) right triangle and have one fourth of its area. The solution with squares is different from the solution in figure 5a.

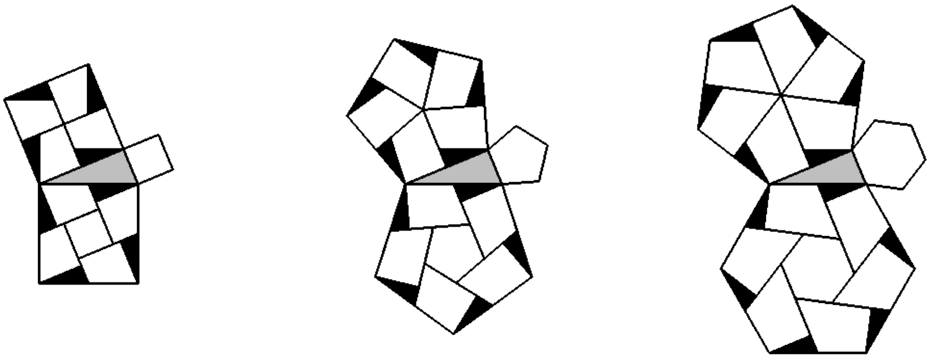

Figure 6. Squares, pentagons, and hexagons

About dissections see (Frederickson, 1997), (Frederickson, 2002), and (Lindgren, 1972).

Teacher's triangle

In the teacher's triangle, i.e. in the well-known Pythagorean right triangle with sides 3, 4, and 5, we will find an easy dissection into 9, 16, and 25 equilateral triangles (Fig. 7a). The only challenge is to color it symmetrically (Fig. 7b).

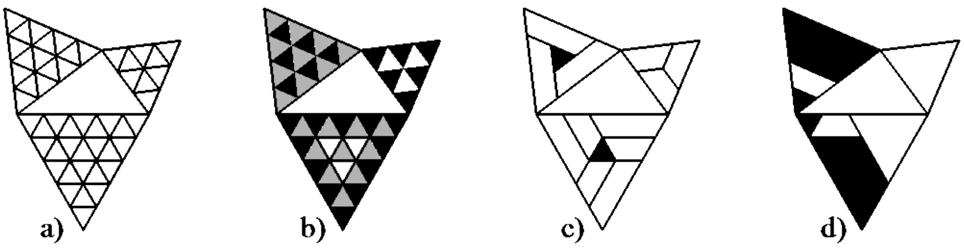

Figure 7. Triangles, triangles, triangles

The dissection of figure 7c has only seven parts and a three-fold symmetry. In figure 7d we need only four parts, but there is no symmetry in the big triangle.
Figure 8 depicts a swing-hingeable dissection from the square to an equal area equilateral triangle. We see the initial square, three intermediate situations and the final equilateral triangle. This example is due to Dudeney (1903) [1].

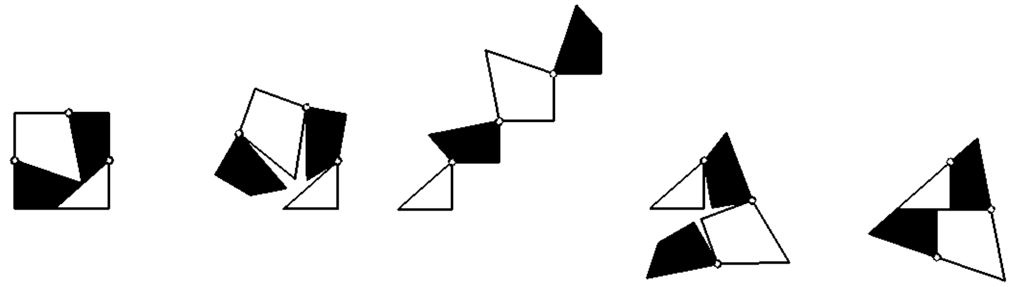

Figure 8. Hinged dissection of a square to a triangle

Two of the four parts (white in figure 8) can be moved by a translation from the initial square to the final equilateral triangle. For the other two parts (black in figure 8) we need a rotation about 180°. The sides of the initial squares are horizontal and vertical with respect to the sides of the paper, but the sides of the final equilateral triangle are not horizontal or vertical.

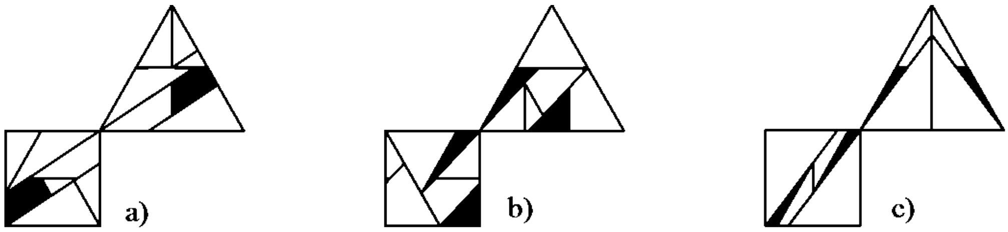

Figure 9. Dissections

In the examples of the figure 9 one side of the equilateral triangle is on a straight line with a side of the square. The reader is kindly invited to accomplish the coloring with appropriate different colors. In the example of figure 9a we can move every part either by a translation or a rotation about 180° from the square to the triangle. In figure 9b we have to rotate only the small equilateral triangle. In figure 9c we need translations and glide reflections to move the parts from the square to the triangle. The dissection of the square has a central symmetry, the dissection of the equilateral triangle an axial symmetry.

General method

Two triangles with the same base and equal height have the same area (Fig. 10a).

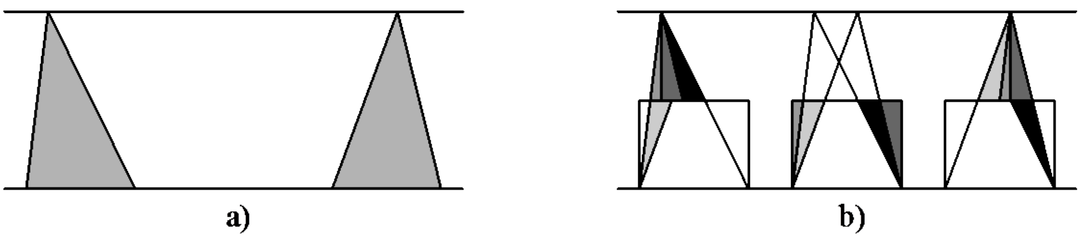

Figure 10. Same area

According to the theorem of Wallace, Bolyai, and Gerwien there must be a common dissection (Fig. 10b). To find the common dissection we need a rectangle as intermediate link. This is already the clue to the general method. But applying a further step of this method, we have also to deal with the dissections of the previous steps.
Figure 11 gives another dissection for the theorem of Pythagoras (Heitzer and Walser, 2014). Again the reader is invited to color corresponding parts.

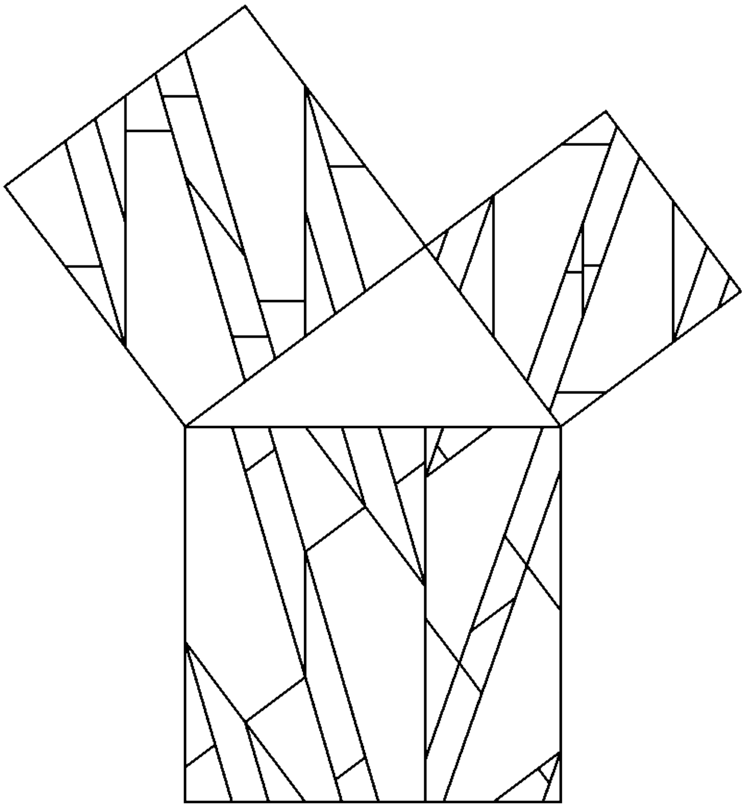

Figure 11. Theorem of Pythagoras

In the dissection of figure 11 corresponding parts may always be moved by translations. The dissections in the small squares have central symmetry, and the dissections in the corresponding rectangles in the large square have central symmetry as well.

Number of colors

The four-color theorem states that any dissection of the plane in different regions can be colored by no more than four colors (Appel and Haken, 1976). But if we have a common dissection of two polygons, we may need more than four colors. In the common dissection of the two polygons of the figure 12 we need for each of the five parts a separate color.

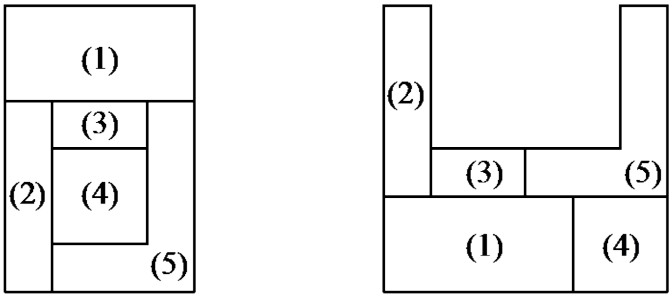

Figure 12. Five colors required

Of course in the polygon on the left we could replace color (4) by color (1), but in the polygon on the right we cannot do so.

Equilateral triangle and regular hexagon

The triangle and the hexagon in figure 13 have the same area.

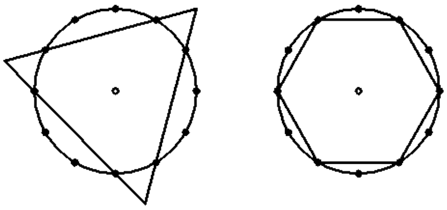

Figure 13. Equal areas

The figure 14 gives two different common dissections. The common dissection of figure 14a requires 9 parts, but only 2 shapes if one counts mirror-imaged triangles as one shape. Otherwise we have 3 shapes. Both dissections have a three-fold rotational symmetry.

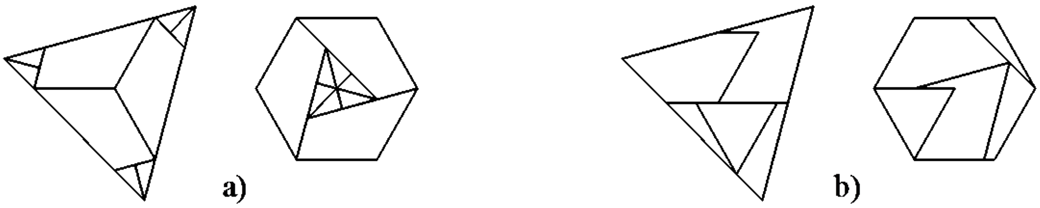

Figure 14. Common dissections

The dissection of figure 14b is the best known minimal solution [1]. It requires five parts only. But the five parts have different shapes, and there is no symmetry.

A problem by Euclid

Euclid, Elements, Book II, Prop. XI. – Problem: To cut a given finite line such that the rectangle contained by the whole line and one segment may be equal to the square on the other segment (Fig. 15, the given finite line is the base line).

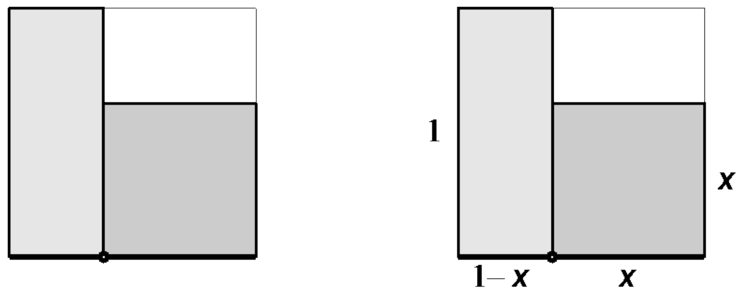

Figure 15. To cut a given finite line

Before we calculate, let's try with fractions (Fig. 16).

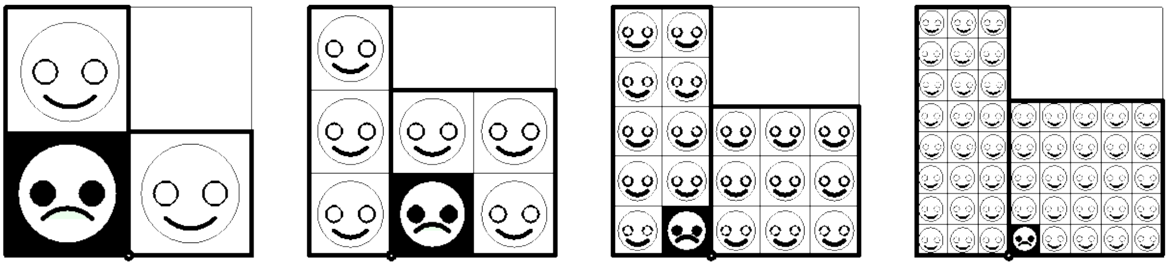

Figure 16. Fractions

The subdivisions $\frac{1}{2}:\frac{1}{2}$, $\frac{1}{3}:\frac{2}{3}$, $\frac{2}{5}:\frac{3}{5}$, $\frac{3}{8}:\frac{5}{8}$, and so on don't solve the problem, but the error (which flips always from one side to the other) becomes smaller and smaller. We recognize the Fibonacci numbers 1, 1, 2, 3, 5, 8, ... (Posamentier and Lehmann, 2007), (Walser, 2012). Therefore it is not surprising that the solution is the Golden Section (Posamentier and Lehmann, 2012), (Walser, 2001), (Walser, 2013)).

$$x = \frac{\sqrt{5}-1}{2} \approx 0.618$$

To find a common dissection of the rectangle and the square we try first the "greedy algorithm": we remove always the largest rectangle fitting in both the original rectangle and the square. Figure 17 depicts the first steps.

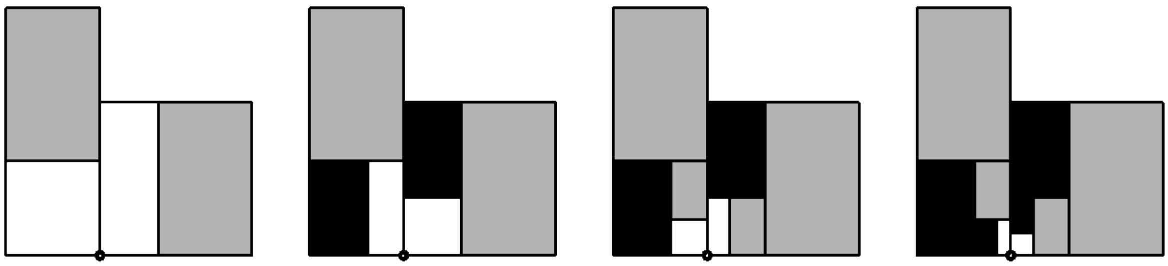

Figure 17. Remove the largest possible common rectangle

But after the first two steps the leftovers, i. e. the remaining parts of the original rectangle and square, are similar to the original figure. Therefore, going on with our algorithm, we will never come to an end. We have a funnel symmetry.

Nevertheless there is a dissection from the rectangle to the square with only four parts and two shapes (Fig. 18). There is a central symmetry both in the dissected rectangle and the square.

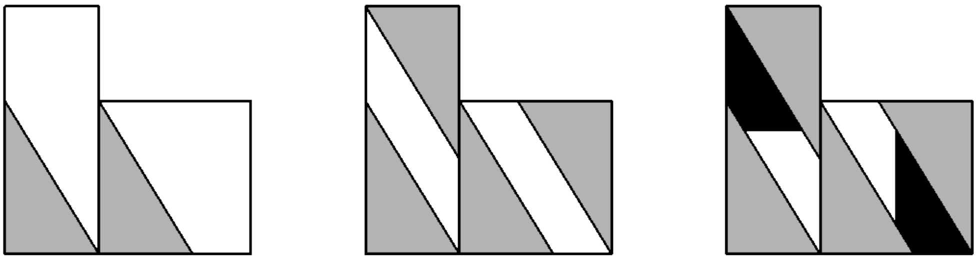

Figure 19. Common dissection

Conclusion

A rigorous proof of area equivalence of two figures is usually done by calculation of the areas. If we work with common dissections, we have to prove that the corresponding parts of the two dissections are congruent and that the parts fit without holes or overlapping in both dissections. The figure 19 depicts a well-known counter example.

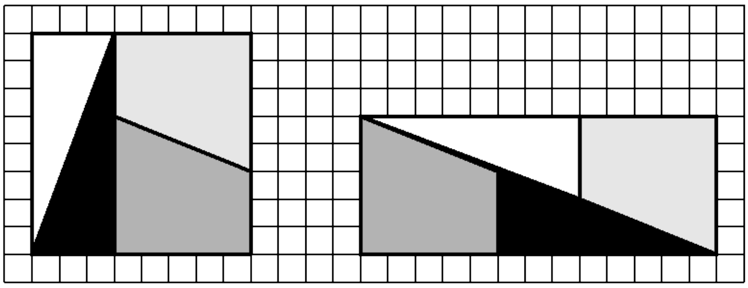

Figure 19. The square and the rectangle are not equivalent

The corresponding parts of the dissections are congruent. The square has the sides 8 and therefore the area 64, but the rectangle has the sides 13 and 5 and hence the larger area 65. The additional unit is due to the fact that the parts do not fit in the rectangle. They leave a hole (in form of a parallelogram) with the area 1.
Therefore using dissections in school does no aim at proofs of theorems as for example the theorem of Pythagoras, but to illustrate and visualize these theorems. It is a question of seeing geometric structures and finally also a question of pure beauty.

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[1] mathworld.wolfram.com/Dissection.html (November 28, 2015)

A MATHEMATICAL EXCURSION

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Abstract: *The mathematician Ian Stewart expressed that we are constantly surrounded by a mathematical spirit. While we usually learn and deal with mathematics in an unfavourable learning environment inside buildings, we tend to forget about the various opportunities given by nature just outside the window. Thus, this article mainly aims at (1) illustrating a concept for the promotion of mathematically gifted and interested pupils in reference to popular theories about this issue and at (2) investigating the positive effects of a mathematical excursion on the development of mathematically gifted and interested pupils. By taking part in a competition such as the mathematical excursion through Schwäbisch Gmünd, pupils are given the opportunity of looking at its buildings, people, etc., in different ways - historically, commercially and mathematically.*

„Curiosity in children is but an appetite after knowledge, and therefore ought to be encouraged in them"

John Locke (1632 – 1704)

In his statement, John Locke, an English philosopher and Enlightenment thinker, refers to the natural curiosity in children and their desire for knowledge. This idea is supported by an increasing body of empirical evidence. In addition, this statement is the source of our intention to encourage students by means of adequate research and keep their desire for knowledge alive. Our research aims at elaborating and testing possibilities to support and promote mathematically talented students. A special form of organisation, a "mathematical excursion" forms the basis of our concept.

What is a mathematical excursion?

"The mathematical excursion leads through the city of the students. It consists of different stations where tasks and problems have to be solved" (see Pürckhauer 2007, p. 59). "The mathematical path through my city" is based on a teaching concept elaborated and tested under the control of Professor Astrid Beckmann (see Beckmann 2002, 2007). Since 2003, numerous academic papers on different research approaches relating to this topic have been published under the direction of Professor Astrid Beckmann at the University of Education Schwäbisch Gmünd (see Jörgens 2007, Pfizter 2007, Pürckhauer 2007 and examples in several cities such as Aalen, Backnang, Bad Mergentheim, Bönnigheim, Eislingen, Esslingen, Geislingen/Steige, Giengen/Brenz, Göppingen, Heidenheim, Köngen, Niedernhall, Schaffhausen-Neuhausen, Schorndorf, Schwäbisch Hall, Spaichingen, Tuttlingen, Waiblingen, as at 2015, on www.mathematischer-weg.ph-gmuend.de).

A concept on the promotion of mathematically talented students

Target group

The majority of support programmes for mathematic talents are geared towards students of the primary and upper secondary level. This is due to both the recommendation of specialist literature to start promoting talents as early as possible and to the offers of mathematical university institutes for high-school graduates. Thus, in keeping with the spirit of universal promotion, our concept focuses on students in the lower secondary level. Being interested in the respective subject is one of the most essential requirements for successful learning (see Murayama 2013, Willems 2011, Klimova 2014).
As the concept focuses on promoting talented and interested students, the respective teachers are responsible for selecting the students. It is recommended to form groups of four to five students for the excursion. It should be ensured that the groups consist of students of different age and gender to motivate cross age and gender learning.

Team competition

Our reflections concerning the elaboration of this concept are based on the person-object approach to interest and multi-dimensional models of giftedness, including Renzulli's three-ring model of giftedness and the Munich model on intellectual giftedness. Findings on the development of a concept promoting talent and interest generally consider team competition an adequate means. This is due to the facts that team competition addresses the mental, physical and socio-cultural requirements of talent and interest and can help to meet certain demands in terms of promotion. In addition, boys generally prefer individual competition while girls rather have a preference for team competition. Thus, the concept of a mathematical excursion based on team competition is an adequate tool to not only promote mathematical talent in male but also in female students.

Tasks

In addition, the concept is oriented towards the attributes of mathematical talent proven by theories and studies. The following table includes the most essential aspects of tasks essential for the promotion of mathematical talent and interest.

Attributes of tasks promoting interest	Attributes of tasks promoting talent
Emotional and value-oriented character	Motivating and creative character
Possibility to identify with the task	*Open tasks*
Promotion of independent thinking	Appropriate level of tasks
Differentiation of situational and individual interests	*Challenging character of tasks*
Consideration of age in terms of centre of interest	Consideration of creative, social and psychomotor aspects
	Different solutions possible
Promotion of activity	Relation to established knowledge

The tasks are supposed to encourage communication within the group as well as argumentation and the discussion of different solutions. In addition, participants are required to develop appropriate strategies to solve the problems given. As the selected tasks refer to everyday situations that have to be transferred to mathematical problems, a modelling process is stimulated.

Organisational matters

The groups must solve the tasks given at the different stations within a certain time frame amounting to two hours in general. However, no detailed solution but rather measurements and problem-solving approaches have to be provided within the given time frame. These first results serve as a basis for subsequent calculations and precise solutions. The groups are provided with an additional time frame of one to two hours, depending on the type of tasks and the level of difficulty, in order to deal with the matter in a detailed manner. Beyond being able to mathematically solve a task, mathematical understanding also includes the ability to verify the plausibility of solutions. The mathematical excursion is topped off with an award ceremony where all participants receive a certificate and the winning group receives mathematical prices. Clear criteria for the evaluation of solutions are determined beforehand. These criteria do not only take into consideration purely mathematical solutions but also the procedure of solution and the manner of representation. Particularly participants assigned to the group of underachievers highly benefit from this approach.

The concept and general conditions

The most important features of our concept on the promotion of mathematically talented students based on the idea of a mathematical path are:

- specific requirements for the tasks,
- team competition as a social form,
- an additional phase for the development of detailed solutions for the tasks.

The theoretical discussion of the concept is followed by an exploratory investigation aiming at elaborating an adequate correlation hypothesis on the following research question:

Does the mathematical excursion have positive impacts on the promotion of the mathematical talent of students attending the lower secondary level?

The concept of a mathematical excursion for talented students was discussed during

- the mathematical weekend for participants from Germany and Switzerland (2013 n=72, 2014 n=66)
- the student exchange between Germany and Ukraine (2014 n=19)

at the Landesgymnasium für Hochbegabte in the Universitätspark Schwäbisch Gmünd.

Selected tasks

Different tours have been prepared for the mathematical excursions in Schwäbisch Gmünd. One of these tours focuses on the city's towers ("From tower to tower") while another tour stands under the motto "Numbers everywhere". The participants of the tower tour work with six of the still preserved town towers, unique sights dating back to more than eight centuries that make up the architectural treasure of the oldest Staufer town. Several tasks of the different stations are presented below *(see next page)*:

Station The Art of Measuring

Station Route Planning

This station enables students to discover sub-areas of mathematics that are not included in maths classes.

Route Planning

Imagine you are tourists in Schwäbisch Gmünd and would like to visit five sights. Of course, you will want to save time and find the shortest distance possible while only visiting every sight once.

Task

a) Choose five random sights including the market place in Schwäbisch Gmünd and draw the respective labeled graph. You will have to decide which connecting roads you will use and mark them in your solution. The market place is both the starting and the end point.

b) Try to find the shortest tour possible with the market place being both the starting and the end point. Developdifferent strategies and compare the results (e.g. you always visit the sight which is closest to the sight you are currently at). Which is the best strategy?

Brief excursion into graph theory

Definition: A structure of endpoints and lines is called graph if every line connects exactly two endpoints. In mathematics, we call the lines edges and the points vertices.

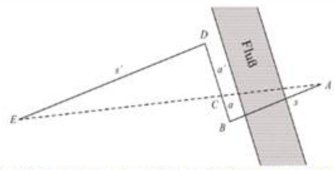
Definition: A graph with length specifications is called a labeled graph.

The Art of Measuring

It is often impossible to directly measure distances. In these cases, however, methods of indirect distance measurement can be applied and include different mathematical theorems such as the Pythagorean theorem, intercept theorems and the similarity of triangles.

You can find numerous waters, flowerbeds or fountains which cannot be directly measured in Schwäbisch Gmünd.

The following sketch can help you to determine the width of a river, a flowerbed, etc. A right angle is built in point B in order to determine a line on which point C is located. The exact location of point C on this line can be freely determined. The line segment from B to C is then extended to any length (point D). Another right angle is built in point D in order to determine point E in alignment with A and C.



Task Choose a river, a flowerbed, a fountain or the like. State the place you chose in your solution!

a) Which theorem is behind this method of distance measurement?

b) Express the theorem verbally and mathematically correct.

c) How can distance s be determined theoretically by using a)? Formulate an algebraic equation!

d) Use the measurement of your distance and calculate the width!

Research findings

Exploratory studies include qualitative and quantitative studies. Qualitative studies comprise an interview with several persons based on certain guidelines and a case-by-case-analysis of the solutions. Quantitative studies, however, are based on a questionnaire geared towards the participants of the mathematical excursion.

Subsequent to the mathematical excursion in 2013, interviews were conducted with eight student assistants. Their statements are summarized in the following table.

Attributes	Does not apply at all	Does not apply	Neutral	Does apply	Does absolutely apply
A: Promotion of independent thinking				**5, 7**	**1, 3, 4, 6, 8**
B: Promotion of activity		**8**	**3**	**1, 2, 6**	**2, 4, 7**
C: Motivating and creative character of the mathematical excursion			**5**	**1, 6**	**2, 3, 4, 7**
D: Challenging character with regard to the lower grades (adequate level of difficulty)		**7, 8**	**1,2, (3)**	**4, 5, 6**	
E: Challenging character with regard to the upper grades (adequate level of difficulty)			**1,2, (3), 8**	**4, 5, 6**	**7**
F: Ability to cooperate within the groups			**2, 5, 7**	**1, 3, 6**	**4, 8**

Figure 1. Characteristics in the context of the interviews

In addition, the quantitative study is based on a cross-sectional study enabling a hypothesis to be generated by means of valence and intensity analyses. The attributes of talent and interest are evaluated by the participants themselves.

The variable of mathematical talent and interest is being allocated attributes that are part of the questionnaire at the personal, group- and content-related level. The attributes of the three levels are explained below.

- Personal attributes: Possibility to identify with the tasks, level of difficulty, numerous possible solutions, mathematical fantasy, motivating character of the tasks
- Group-related attributes: Ability to cooperate, discussion about mathematical issues, transitive transfer of interests, "cross-age-learning"
- Content-related attributes: Relation to established mathematical contents, generation of new mathematical knowledge on the basis of established knowledge, relation between mathematics and reality with regard to the emotional and value-related character of the tasks.

In the context of these attributes, the mathematical excursion was assessed positively (see figure 2). The tasks in 2014 were considered a challenge for the participants (6.4, 8.6, 6.2, 9.2), served as a basis for discussion and reflection (8.3, 9.1) and had positive psychological impacts (6.1, 9.3). The mathematical excursion promoted the cooperative way of working (9.10) and provided participants with insights in different fields of application of mathematics (6.3, 8.8).

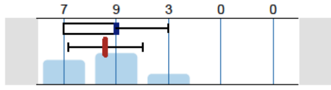
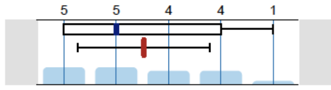
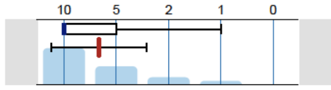
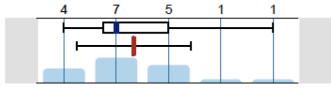
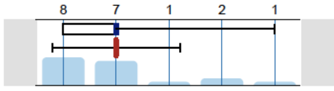
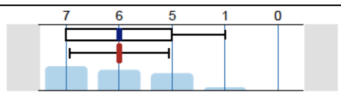
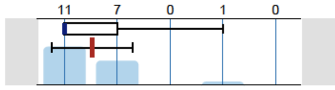
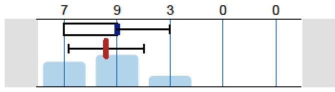
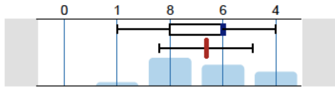
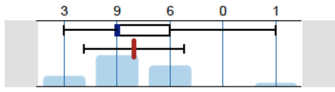
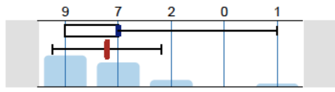
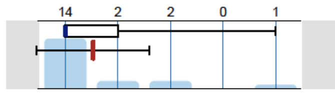
6.1	I had a good time during the mathematical excursion	I fully agree		I disagree	n=19 mw=1.8 md=2 s=0.7
6.2	I have gained new knowledge in the field of mathematics	I fully agree		I disagree	n=19 mw=2.5 md=2 s=1.3
6.3	I have gained insight in different fields of application of mathematics	I fully agree		I disagree	n=18 mw=1.7 md=1 s=0.9
6.4	I have solved challenging mathematical problems	I fully agree		I disagree	n=18 mw=2.3 md=2 s=1.1
8.3	During the mathematical excursion, I have discussed mathematical problems with the other group members	I fully agree		I disagree	n=19 mw=2 md=2 s=1.2
8.6	I had enough knowledge to successfully solve the tasks	I fully agree		I disagree	n=19 mw=2 md=2 s=0.9
8.8	I was able to recognize the links between the mathematical tasks and reality	I fully agree		I disagree	n=19 mw=1.5 md=1 s=0.8
9.1	I was given food for thought during the mathematical excursion	I fully agree		I disagree	n=19 mw=1.8 md=2 s=0.7
9.2	The tasks were too complex	I fully agree		I disagree	n=19 mw=3.7 md=4 s=0.9
9.3	I have enjoyed searching for solutions	I fully agree		I disagree	n=19 mw=2.3 md=2 s=0.9
9.10	Cooperating with the other group members was important to me	I fully agree		I disagree	n=19 mw=1.8 md=2 s=1
10.5	The mathematical excursion is a good addition to regular classes	I fully agree		I disagree	n=19 mw=1.5 md=1 s=1.1

Figure 2. Questionnaire analysis

Due to a lack of space, the single research findings can only be discussed to a limited extent. In general, the participants consider the mathematical excursion a good supplementation to regular maths classes (10.5).

Conclusion

Based on the results of the qualitative and quantitative empirical studies, the following research hypothesis on the potential for promotion of the mathematical excursion can be generated and adequate modifications for the concept can be proposed.

The participation in a mathematical excursion and the promotion of mathematical talent and interest show a positive correlation.

Modification proposals are to be made for the future development of the discussed concept in order to support this hypothesis. The personal, group- and content-related levels are used as a basic structure. In terms of personal attributes, a larger possibility to identify with the problems is aimed at. In addition, age-related considerations will be integrated in the planning in order to better tailor the different tours to the physical conditions of the respective age groups. In addition, the group assignment will have to be reconsidered. A positive conclusion can be drawn as the basic structure of the concept can be maintained with minor modifications.
When abstracting the findings gained, a positive correlation between participating in a mathematical excursion and successfully promoting mathematical talent and interest is revealed.

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